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Spectral analysis of coupled linear complementarity problems

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ABSTRACT

This note deals with the so-called cone-constrained bivariate eigenvalue problem. The equilibrium model under consideration is a system of linear complementarity problems

$$\begin{cases} P \ni x \perp (Ax + By - \lambda x) \in P^*, \\ Q \ni y \perp (Cx + Dy - \mu y) \in Q^* \end{cases}$$

involving two closed convex cones and their corresponding duals. We study the set of pairs $(\lambda, \mu) \in \mathbb{R}^2$ for which this system has a "nontrivial" solution $(x, y) \in \mathbb{R}^{n+m}$. We discuss also the link between the cone-constrained version and the unconstrained one.

1. Introduction

1.1. The classical setting

The classical bivariate eigenvalue problem consists in finding a pair (λ, μ) of real numbers such that the system of linear equations

$$\begin{cases} Ax + By = \lambda x, \\ Cx + Dy = \mu y \end{cases} \tag{1}$$

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has a solution $(x, y) \in \mathbb{R}^{n+m}$ satisfying the double normalization condition

$$\|x\| = 1, \quad \|y\| = 1.$$

Such a pair $(\lambda, \mu) \in \mathbb{R}^2$ is called a *strong bi-eigenvalue* of the block structured matrix

$$E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

The blocks of E are real matrices of appropriate size, namely $A \in \mathbb{M}_n,\ D \in \mathbb{M}_m,\ B \in \mathbb{M}_{n,m}$, and $C \in \mathbb{M}_{m,n}$. If B and C are zero matrices, then the system (1) unfold into two separate standard eigenvalue problems; otherwise, B and C induce a coupling between the state vectors x and y.

Bivariate eigenvalue problems arise in various fields, but surprisingly the theoretical literature on the subject is not so extensive after all. Perhaps the earliest publication introducing a concrete bivariate eigenvalue problem was a 1935 paper by Hotelling [18]. The specific problem treated by Hotelling concerns the determination of canonical correlation coefficients for bivariate statistics; see also [17,22]. An iterative method for solving bivariate eigenvalue problems was proposed in 1961 by Horst [16]. The convergence of Horst's iterative method was proved three decades later by Chu and Watterson [9]. Among a few recent contributions to the theory of bivariate eigenvalue problems we cite the papers by Hanafi and Ten Berge [15], Barkmeijer and van Noorden [3], Chu and Zhang [10], and Liu et al. [24].

For notational simplicity we stick to the bivariate case, but several of our results can be formulated in a multivariate setting. The p-variate version of (1) consists in finding a p-tuple $(\lambda_1, \ldots, \lambda_p)$ of real numbers such that

$$\begin{bmatrix} K_{1,1} & K_{1,2} & \ldots & K_{1,p} \\ K_{2,1} & K_{2,2} & \ldots & K_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ K_{p,1} & K_{p,2} & \ldots & K_{p,p} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_p \end{bmatrix} = \begin{bmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \\ \vdots \\ \lambda_p z_p \end{bmatrix}$$

has a solution $(z_1, z_2, \ldots, z_p) \in \mathbb{R}^{n_1+\ldots+n_p}$ satisfying the p -fold normalization condition

$$\|z_1\| = 1,\ \|z_2\| = 1,\ \ldots, \|z_p\| = 1.$$

Remark 1.1. Despite an almost identical name, the bivariate eigenvalue problem (1) is fundamentally different from the so-called two-parameter eigenvalue problem. The later one consists in finding a pair $(\lambda, \mu) \in \mathbb{R}^2$ such that

$$\begin{cases} T_1 x = \lambda R_1 x + \mu S_1 x, \\ T_2 y = \lambda R_2 y + \mu S_2 y \end{cases}$$

has a nontrivial solution $(x, y) \in \mathbb{R}^{n+m}$. General information on the two-parameter model can be found in [8,26] and references therein.

1.2. The cone-constrained version

The bivariate eigenvalue problem addressed in this paper is a generalization of (1). In our work, the state vector x is further restricted by means of a nonzero closed convex cone P in $\mathbb{R}^n$. Roughly speaking, P serves to model a possibly infinite number of linear inequality constraints. Similarly, the state vector y is further restricted by means of a nonzero closed convex cone Q in $\mathbb{R}^m$. Instead of a system of linear equations like in (1), the equilibrium model under consideration is now a system of linear complementarity problems:

$$\begin{cases} P \ni x \perp (Ax + By - \lambda x) \in P^*, \\ Q \ni y \perp (Cx + Dy - \mu y) \in Q^*. \end{cases} \tag{2}$$

As usual, the symbol "$\perp$" indicates orthogonality in the appropriate Euclidean space. For instance, $u \perp x$ if and only if $u^T x = 0$ with "T" denoting transposition. The sets

$$P^* = \{u \in \mathbb{R}^n : u^T x \geqslant 0 \text{ for all } x \in P\},$$
$$Q^* = \{v \in \mathbb{R}^m : v^T y \geqslant 0 \text{ for all } y \in Q\}$$

are the dual cones of P and Q, respectively.

The model (2) is introduced here for the first time, but there are some variants disseminated in the literature. Note that (2) can be reformulated as a variational inequality, namely

$$\begin{cases} (x, y) \in P \times Q \quad \text{and} \\ (Ax + By - \lambda x)^T (x' - x) + (Cx + Dy - \mu y)^T (y' - y) \geqslant 0 \quad \text{for all } (x', y') \in P \times Q. \end{cases}$$

Viewed under this light, (2) bears a resemblance to a number of equilibrium models in nonsmooth mechanics discussed by Bocea et al. [4–7]. However, the work of these authors is very different in spirit from our.

We mention in passing that the single linear complementarity problem

$$P \ni x \perp (Ax - \lambda x) \in P^*$$

has drawn a great deal of attention in the last decade; see, for instance, [13,14,19,28] and the contributions of Seeger and collaborators [23,29,30]. We shall take advantage of the knowledge gained in the single case, but we shall also need to innovate in order to cope with the coupling phenomenon.

That (1) is a particular case of (2) is clear: it suffices to take $P = \mathbb{R}^n$ and $Q = \mathbb{R}^m$. There are good reasons for studying a cone-constrained bivariate eigenvalue problem like (2). Imagine for a moment that one needs to solve a minimization problem

$$\begin{aligned} \text{minimize} \quad & x^T U x + y^T W x + y^T V y \\ & x \in P, \quad y \in Q, \\ & \|x\| = 1, \quad \|y\| = 1, \end{aligned} \tag{3}$$

with a coupling effect on the vectors x and y induced by a certain matrix W. If one works out the stationarity or first-order optimality conditions for (3), then, in addition to the double normalization requirement, one obtains the system (2) with

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{U+U^T}{2} & \frac{W^T}{2} \\ \frac{W}{2} & \frac{V+V^T}{2} \end{bmatrix}. \tag{4}$$

The details will be explained in the proof of Theorem 2.1. For the time being, suffice it to say that the optimization literature furnishes a long list of interesting variational problems that can be cast in the abstract framework (3). Just two examples will do as a start:

Example 1.2. Let P be a nonzero closed convex cone in some Euclidean space, say $\mathbb{R}^n$. The maximal angle of P, denoted by $\theta_{\max}(P)$, is the largest angle that can be formed by picking a pair of unit vectors from P. In other words,

$$\cos[\theta_{\max}(P)] = \min_{\substack{x,y \in P \\ \|x\|=1,\ \|y\|=1}} y^T x. \tag{5}$$

This is a special instance of the minimization problem (3). Iusem and Seeger [20,21] have developed a theory of critical angles for convex cones that is based on the analysis of the bi-eigenvalue problem associated to (5).

Example 1.3. The concept of minimal angle (or smallest canonical angle) between two linear subspaces of $\mathbb{R}^n$ has been considered by a number of authors, see for instance the book by Meyer [25, Section 5.15]. The definition of minimal angle extends in a natural way to a pair of nonzero closed convex cones in $\mathbb{R}^n$, say P and Q. The minimal angle between P and Q, denoted by $\theta_{\min}(P, Q)$, is given by

$$\cos[\theta_{\min}(P, Q)] = \max_{\substack{x \in P, y \in Q \\ \|x\|=1,\ \|y\|=1}} y^T x. \tag{6}$$

Multiplying by -1 on both sides of (6) one recovers yet another minimization problem that fits into the model (3).

1.3. Comments on terminology

In fact, we distinguish between two sorts of bi-eigenvalues. Everything depends on whether one wishes a solution to (2) to be a vector on the bi-sphere

$$\mathbb{S}_{n,m} = \{(x, y) \in \mathbb{R}^{n+m} : \|x\| = 1, \ \|y\| = 1\}$$

or a vector in the larger set

$$\mathbb{T}_{n,m} = \{(x, y) \in \mathbb{R}^{n+m} : x \neq 0, \ y \neq 0\}.$$

Each option has its own advantages and inconveniences. In general, the sets

$$\mathcal{S}(E, P, Q) = \left\{(\lambda, \mu) \in \mathbb{R}^2 : (2) \text{ has a solution in } \mathbb{S}_{n,m}\right\},$$
$$\mathcal{T}(E, P, Q) = \left\{(\lambda, \mu) \in \mathbb{R}^2 : (2) \text{ has a solution in } \mathbb{T}_{n,m}\right\}$$

are different. We refer to the first set (respectively, the second set) as the *strong bi-spectrum* (respectively, the *weak bi-spectrum*) of the system (2). Needless to say, one always has the inclusion

$$\mathcal{S}(E, P, Q) \subset \mathcal{T}(E, P, Q).$$

The notation that is being used emphasizes the role played by the cones P and Q. When dealing with a classical or unconstrained bi-eigenvalue problem, one simply writes

$$\mathcal{S}(E) = \left\{(\lambda, \mu) \in \mathbb{R}^2 : (1) \text{ has a solution in } \mathbb{S}_{n,m}\right\},$$

$$\mathcal{T}(E) = \left\{(\lambda, \mu) \in \mathbb{R}^2 : (1) \text{ has a solution in } \mathbb{T}_{n,m}\right\}.$$

The later sets are respectively the strong bi-spectrum and the weak bi-spectrum of E.

An important point to be kept in mind is that $\mathcal{S}(E, P, Q)$ could have infinitely many elements. As illustrated in the next example, the same remark applies to $\mathcal{S}(E)$.

Example 1.4. The strong bi-spectrum of the structured matrix

$$\begin{bmatrix} A & 0 \\ c^T & d \end{bmatrix}$$

is not finite if the block A admits a real eigenvalue λ of geometric multiplicity greater than one, and such that c is not orthogonal to the corresponding eigenspace $\mathrm{Ker}(A - \lambda I_n)$. To be more down to earth, consider $n = 3$, $m = 1$, and the matrix

$$\begin{bmatrix} 7 & 0 & 2 & 0 \\ 0 & 7 & 2 & 0 \\ 0 & 0 & 8 & 0 \\ 3 & 4 & 4 & 1 \end{bmatrix}.$$

Its strong bi-spectrum is formed by the line segment $\{7\} \times [-4, 6]$ and the isolated points $(8, -5)$ and $(8, 7)$.

Topologically speaking, strong and weak bi-spectra are very different in nature. Two comments are useful to put things in the right perspective. Firstly, $\mathcal{S}(E, P, Q)$ is always closed. By contrast, $\mathcal{T}(E, P, Q)$ may not be closed. A weak bi-spectrum is often a finite union of smooth curves on which some points are missing. And, secondly, $\mathcal{S}(E, P, Q)$ is always bounded. By contrast, $\mathcal{T}(E, P, Q)$ is often unbounded.

The next proposition shows that $\mathcal{T}(E, P, Q)$ remains the same if one replaces $\mathbb{T}_{n,m}$ by any of the following normalization sets:

$$\mathbb{T}^{\mathrm{a}}_{n,m} = \{(x, y) \in \mathbb{R}^{n+m} : \|x\| \geqslant 1, \ \|y\| \geqslant 1\}, \tag{7}$$

$$\mathbb{T}^{\mathrm{b}}_{n,m} = \{(x, y) \in \mathbb{R}^{n+m} : 0 < \|x\| \leqslant 1, \ \ 0 < \|y\| \leqslant 1\}. \tag{8}$$

Such a proposition is of interest when it comes to deal with convergence issues: one can use (8) if one wishes x and y to stay uniformly **a**way from the origin, and (7) if one needs to make sure that (x, y) stay in a bounded set. For economy of words, we use the notation $\Xi(\mathbb{R}^n)$ for indicating the collection of nonzero closed convex cones in $\mathbb{R}^n$.

Proposition 1.5. *Let $P \in \Xi(\mathbb{R}^n)$ and $Q \in \Xi(\mathbb{R}^m)$. Then,*

$$\mathcal{T}(E, P, Q) = \left\{(\lambda, \mu) \in \mathbb{R}^2 : (2) \text{ has a solution in } \mathbb{T}^{\mathrm{a}}_{n,m}\right\} \tag{9}$$

$$= \left\{(\lambda, \mu) \in \mathbb{R}^2 : (2) \text{ has a solution in } \mathbb{T}^{\mathrm{b}}_{n,m}\right\}. \tag{10}$$

Proof. Let $(\lambda, \mu) \in \mathbb{R}^2$. If $(x, y) \in \mathbb{T}_{n,m}$ solves (2), then also the pairs

$$(x', y') = \left(\frac{x}{\min\{\|x\|, \|y\|\}}, \frac{y}{\min\{\|x\|, \|y\|\}}\right) \in \mathbb{T}^{\mathrm{a}}_{n,m},$$
$$(x'', y'') = \left(\frac{x}{\max\{\|x\|, \|y\|\}}, \frac{y}{\max\{\|x\|, \|y\|\}}\right) \in \mathbb{T}^{\mathrm{b}}_{n,m}$$

solve (2). This proves the relation "$\subset$" in (9) and (10). The reverse inclusions are obvious because $\mathbb{T}^{\mathrm{a}}_{n,m}$ and $\mathbb{T}^{\mathrm{b}}_{n,m}$ are both contained in $\mathbb{T}_{n,m}$. $\square$

2. Existence of bi-eigenvalues

2.1. The symmetric case

This section addresses the issue of existence of bi-eigenvalues in a cone-constrained setting. The next theorem comes without surprise. It concerns the case in which (2) derives from a minimization problem.

Theorem 2.1. *Let $P \in \Xi(\mathbb{R}^n)$ and $Q \in \Xi(\mathbb{R}^m)$. If E is symmetric, then $\mathcal{S}(E, P, Q)$ is nonempty.*

Proof. Since the block structured matrix E is symmetric, it can be represented as in (4). In other words, the quadratic form associated to E is nothing but the cost function of (3). By compactness, the minimization problem (3) admits at least one solution, say $(\bar{x}, \bar{y}) \in (P \times Q) \cap \mathbb{S}_{n,m}$. A standard argument of optimization theory ensures the existence of a pair $(\lambda, \mu) \in \mathbb{R}^2$ of Lagrange multipliers such that $(\bar{x}, \bar{y})$ satisfies the stationarity condition

$$P \ni \bar{x} \perp \nabla_x L(\bar{x}, \bar{y}, \lambda, \mu) \in P^*,$$
$$Q \ni \bar{y} \perp \nabla_y L(\bar{x}, \bar{y}, \lambda, \mu) \in Q^*$$

with $L : \mathbb{R}^{n+m+2} \to \mathbb{R}$ denoting the Lagrangean function given by

$$L(x, y, \lambda, \mu) = x^T Ux + y^T Wx + y^T Vy - \lambda \left(\|x\|^2 - 1\right) - \mu \left(\|y\|^2 - 1\right).$$

For completing the proof it suffices to observe that

$$\nabla_x L(\bar{x}, \bar{y}, \lambda, \mu) = (U + U^T)\bar{x} + W^T\bar{y} - 2\lambda\bar{x},$$
$$\nabla_y L(\bar{x}, \bar{y}, \lambda, \mu) = W\bar{x} + (V + V^T)\bar{y} - 2\mu\bar{y}.$$

One sees that $(\lambda, \mu) \in \mathcal{S}(E, P, Q)$ because (2) has $(\bar{x}, \bar{y})$ as double normalized solution. □

The formula (11) given below enhances the importance of the strong bi-spectrum as mathematical tool for dealing with symmetric structured matrices. A far ancestor of (11) is the celebrated Raleigh–Ritz principle, according to which the smallest eigenvalue of a symmetric matrix is equal to the infimal value on the unit sphere of the associated quadratic form.

Proposition 2.2. *Let $P \in \Xi(\mathbb{R}^n)$ and $Q \in \Xi(\mathbb{R}^m)$. If E is given by (4) and γ denotes the infimal value of the minimization problem (3), then*

$$\gamma = \min\{\lambda + \mu : (\lambda, \mu) \in \mathcal{S}(E, P, Q)\}. \tag{11}$$

Proof. Let $\tilde{\gamma}$ be the term on the right-hand side of (11). The minimum defining $\tilde{\gamma}$ is attained because $\mathcal{S}(E, P, Q)$ is nonempty and compact. Let $(\lambda, \mu) \in \mathcal{S}(E, P, Q)$ be a pair achieving this minimum, and let (x, y) be a double normalized solution to (2). Then

$$\begin{aligned}\tilde{\gamma} = \lambda + \mu &= x^T(Ax + By) + y^T(Cx + Dy)\\ &= x^T Ux + y^T Wx + y^T Vy.\end{aligned}$$

This proves $\tilde{\gamma} \geqslant \gamma$. Conversely, let $(\bar{x}, \bar{y}) \in (P \times Q) \cap \mathbb{S}_{n,m}$ be a solution to (3), and let λ, μ be associated Lagrange multipliers as in the proof of Theorem 2.1. Then $(\lambda, \mu) \in \mathcal{S}(E, P, Q)$ and

$$\begin{aligned}\gamma &= \bar{x}^T U\bar{x} + \bar{y}^T W\bar{x} + \bar{y}^T V\bar{y}\\ &= \bar{x}^T(A\bar{x} + B\bar{y}) + \bar{y}^T(C\bar{x} + D\bar{y}) = \lambda + \mu.\end{aligned}$$

This proves the reverse inequality $\gamma \geqslant \tilde{\gamma}$. □

2.2. The asymmetric case

All the applications of the model (2) mentioned in Section 1.2 fall within the context of the symmetric case. It is harder to justify the need of studying the asymmetric case because this would require to enter into a long discussion on the theory of variational inequalities and their role in nonsmooth mechanics. Anyhow, the asymmetric case, undoubtedly more complicated than the symmetric one, is very appealing from a purely mathematical point of view. If E is not symmetric, then it is no longer possible to interpretate (2) as the stationarity condition for a cone-constrained quadratic minimization problem. It is easy to construct an example showing that not just the strong bi-spectrum $\mathcal{S}(E)$, but also the weak bi-spectrum $\mathcal{T}(E)$ can be empty.

Example 2.3. Consider the unconstrained system (1) with

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

The first equation in (1) reduces to $Ax = \lambda x$. Since A does not have real eigenvalues, the vector x must be equal to zero. In short, the weak bi-spectrum of E is empty.

Example 2.4 is somewhat anomalous because if B is a null matrix, then the first equation in (1) is free of the state vector y. As shown in the next example, $\mathcal{T}(E)$ can be empty even if x and y show up in each equation of the system (1).

Example 2.4. For the particular choice

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

the unconstrained system (1) becomes

$$\begin{cases} Ax + y = \lambda x, \\ 2x = \mu y. \end{cases}$$

If this system had a solution $(x, y) \in \mathbb{T}_{2,2}$, then $\mu \neq 0$ and $Ax = (\lambda - 2\mu^{-1})x$. But this is not possible because A does not have real eigenvalues.

In the next proposition the symbols $\sigma(A)$ and $\rho(A)$ stand respectively for the real spectrum and the real resolvent set of A.

Proposition 2.5. *The weak bi-spectrum of E is nonempty under any of the following hypotheses*:

(i) $\mathrm{Ker} B = \{0\}$ *and* $\sigma(D - C(A - \lambda I_n)^{-1}B) \neq \emptyset$ *for some* $\lambda \in \rho(A)$.
(ii) $\mathrm{Ker} C = \{0\}$ *and* $\sigma(A - B(D - \mu I_m)^{-1}C) \neq \emptyset$ *for some* $\mu \in \rho(D)$.

Proof. Consider for instance the hypothesis (i). Let $\lambda \in \rho(A)$ be such that $D - C(A - \lambda I_n)^{-1}B$ admits a real eigenvalue, say μ. Let $y \in \mathbb{R}^m \backslash \{0\}$ be an associated eigenvector, i.e.,

$$(D - C(A - \lambda I_n)^{-1}B)y = \mu y. \tag{12}$$

Since $\mathrm{Ker} B = \{0\}$, also

$$x = -(A - \lambda I_n)^{-1}By \tag{13}$$

is a nonzero vector. Since (x, y) solves the system (1), it follows that $(\lambda, \mu) \in \mathcal{T}(E)$. $\square$

If one wishes, one can normalize the eigenvector y in (12), but there is no guarantee that (13) will also be of unit norm. When E is not symmetric, working with $\mathcal{S}(E)$ could be problematic. The next theorem can be considered as the main result of this section. Recall that a convex cone is said to be *pointed* if it does not contain a line.

Theorem 2.6. *Let E be a block structured matrix, symmetric or not. If* $P \in \Xi(\mathbb{R}^n)$ *and* $Q \in \Xi(\mathbb{R}^m)$ *are pointed, then* $\mathcal{T}(E, P, Q)$ *is nonempty.*

Proof. Since P is pointed, there exists a unit vector u in the interior of P^* such that

$$P_u = \{x \in P : u^T x = 1\} \tag{14}$$

is nonempty and bounded. Similarly, one can find a unit vector v in the interior of Q^* such that

$$Q_v = \{y \in Q : v^T y = 1\} \tag{15}$$

is nonempty and bounded. Of course, P_u and Q_v are also closed and convex. Hence, $\Omega = P_u \times Q_v$ is a nonempty compact convex set contained in $P \times Q$. Let $\Psi : \Omega \times \Omega \to \mathbb{R}$ be defined by

$$\Psi[(x, y), (p, q)] = -\left[p^T z_{x,y} + q^T w_{x,y}\right]$$

with

$$z_{x,y} = Ax + By - \frac{x^T(Ax + By)}{\|x\|^2}\, x,$$

$$w_{x,y} = Cx + Dy - \frac{y^T(Cx + Dy)}{\|y\|^2}\, y.$$

Dividing by $\|x\|^2$ is not a problem because P_u does not contain the origin of $\mathbb{R}^n$. By a similar reason, there is no trouble with the division by $\|y\|^2$. One can easily check that

$$\begin{cases} \text{for all } (p,q) \in \Omega, \ \Psi[(\cdot,\cdot),(p,q)] \text{ is continuous,} \\ \text{for all } (x,y) \in \Omega, \ \Psi[(x,y),(\cdot,\cdot)] \text{ is linear,} \\ \text{for all } (x,y) \in \Omega, \ \Psi[(x,y),(x,y)] = 0. \end{cases}$$

In view of these properties, a celebrated result by Ky Fan (see [2, Theorem 3.1.1] or the original source [12]) ensures the existence of a point $(\bar{x},\bar{y}) \in \Omega$ such that

$$\Psi[(\bar{x},\bar{y}),(p,q)] \leqslant 0 \quad \text{for all } (p,q) \in \Omega.$$

This inequality yields in particular

$$\Psi[(\bar{x},\bar{y}),(p,\bar{y})] \leqslant 0 \quad \text{for all } p \in P_u, \tag{16}$$

$$\Psi[(\bar{x},\bar{y}),(\bar{x},q)] \leqslant 0 \quad \text{for all } q \in Q_v. \tag{17}$$

If one defines

$$\lambda = \frac{\bar{x}^T(A\bar{x}+B\bar{y})}{\|\bar{x}\|^2}, \quad \mu = \frac{\bar{y}^T(C\bar{x}+D\bar{y})}{\|\bar{y}\|^2},$$

then it is clear that

$$P \ni \bar{x} \perp (A\bar{x}+B\bar{y}-\lambda\bar{x}),$$
$$Q \ni \bar{y} \perp (C\bar{x}+D\bar{y}-\mu\bar{y}).$$

On the other hand, (17) and (16) take the form

$$p^T(A\bar{x}+B\bar{y}-\lambda\bar{x}) \geqslant 0 \quad \text{for all } p \in P_u,$$
$$q^T(C\bar{x}+D\bar{y}-\mu\bar{y}) \geqslant 0 \quad \text{for all } q \in Q_v,$$

respectively. By positive homogeneity, the first inequality can be extended from P_u to the whole cone P, and the second inequality can be extended from Q_v to the whole cone Q. This completes the proof. $\square$

Instead of the sets (14) and (15), one could have worked with any other pair of convex bases, for instance,

$$\widehat{P} = \operatorname{co}\{x \in P : \|x\| = 1\},$$
$$\widehat{Q} = \operatorname{co}\{y \in Q : \|y\| = 1\}.$$

As usual, "co" stands for the convex hull operation. There is a bit room for generalization in the formulation of Theorem 2.6, but entering into the details would obscure the overall presentation of our work. The pointedness assumption can be relaxed to some extent, but it cannot be dropped altogether; just think of Example 2.4.

3. Reduction techniques and cardinality issues

3.1. Linearly constrained model

If P is a p-dimensional linear subspace of $\mathbb{R}^n$ and Q is a q-dimensional linear subspace of $\mathbb{R}^m$, then (2) can be converted into an unconstrained bivariate eigenvalue problem. It is enough to write

$$P = \operatorname{Im} R, \quad Q = \operatorname{Im} S \tag{18}$$

for suitable full rank matrices $R \in \mathbb{M}_{n,p}$ and $S \in \mathbb{M}_{m,q}$. By using the Gram–Schmidt orthonormalization method, one can even use a representation

$$P = \operatorname{Im} \widehat{R}, \quad Q = \operatorname{Im} \widehat{S}, \tag{19}$$

with $\widehat{R} \in \mathbb{M}_{n,p}$, $\widehat{S} \in \mathbb{M}_{m,q}$ such that $\widehat{R}^T\widehat{R} = I_p$ and $\widehat{S}^T\widehat{S} = I_q$. A bit of linear algebra shows that

$$\mathcal{T}(E, P, Q) = \mathcal{T}(\widetilde{E}) = \mathcal{T}(\widehat{E}), \tag{20}$$

with

$$\widetilde{E} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} = \begin{bmatrix} (R^TR)^{-1}R^TAR & (R^TR)^{-1}R^TBS \\ (S^TS)^{-1}S^TCR & (S^TS)^{-1}S^TDS \end{bmatrix},$$

$$\widehat{E} = \begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & \widehat{D} \end{bmatrix} = \begin{bmatrix} \widehat{R}^TA\widehat{R} & \widehat{R}^TB\widehat{S} \\ \widehat{S}^TC\widehat{R} & \widehat{S}^TD\widehat{S} \end{bmatrix}. \tag{21}$$

Concerning the strong bi-spectrum, a representation like (18) is not precise enough. One must rely on (19) for making sure that unit vectors are converted into unit vectors. What one gets is

$$\mathcal{S}(E, P, Q) = \mathcal{S}(\widehat{E}).$$

Note that $\widehat{E}$ is a matrix of order $p + q$. By the way, if E is symmetric, then so is $\widehat{E}$.

3.2. The polyhedral case

Next we assume that P and Q are polyhedral cones and explain how to convert (2) into a finite family of unconstrained bivariate eigenvalue problems. The next theorem exploits the fact that a closed convex cone is partitioned by its constituent faces. By a *face* of P one understands a convex cone F, subset of P, such that

$$u, v \in P \quad \text{and} \quad u + v \in P \Longrightarrow u, v \in F.$$

A face is necessarily closed. The collection of all faces of P is denoted by $\mathcal{F}(P)$. Recall that a polyhedral cone has finitely many faces and that the dimension of a face F is simply the dimension of span$F$, the linear space spanned by F. Without further ado, we state:

Proposition 3.1. *If $P \in \Xi(\mathbb{R}^n)$ and $Q \in \Xi(\mathbb{R}^m)$ are polyhedral cones, then*

$$\mathcal{T}(E, P, Q) \subset \bigcup_{F \in \mathcal{F}(P),\, G \in \mathcal{F}(Q)} \mathcal{T}(E, \operatorname{span} F, \operatorname{span} G)\,, \tag{22}$$

$$\mathcal{S}(E, P, Q) \subset \bigcup_{F \in \mathcal{F}(P),\, G \in \mathcal{F}(Q)} \mathcal{S}(E, \operatorname{span} F, \operatorname{span} G)\,. \tag{23}$$

Proof. Take any (λ, μ) in $\mathcal{T}(E, P, Q)$. Let $(x, y) \in \mathbb{T}_{n,m}$ be a solution to (2). The vector x is in the relative interior of some face $F \in \mathcal{F}(P)$ and the vector y is in the relative interior of some face $G \in \mathcal{F}(Q)$. One has

$$x \in F \subset F - F = \operatorname{span} F,$$
$$y \in G \subset G - G = \operatorname{span} G.$$

By applying a similar technique as in [30, Theorem 3.4], one can show that

$$Ax + By - \lambda x \in [\operatorname{span} F]^{\perp},$$
$$Cx + Dy - \mu y \in [\operatorname{span} G]^{\perp}.$$

In short, we have found $F \in \mathcal{F}(P)$ and $G \in \mathcal{F}(Q)$ such that $(\lambda, \mu) \in \mathcal{T}(E, \operatorname{span}F, \operatorname{span}G)$. The proof of (23) is similar. $\square$

For a given pair (F, G) of faces, evaluating $\mathcal{T}(E, \mathrm{span}F, \mathrm{span}G)$ is just a matter of solving an unconstrained bivariate eigenvalue problem. However, one must be aware that the computation of the right-hand side of (22) could be extremely expensive if the polyhedral cones P and Q have a large number of faces. The same remark applies to the computation of the right-hand side of (23). Another point not to be forgotten is that the inclusions (22) and (23) are strict in general.

3.3. Counting algebraic curves

This subsection is rather technical and deviates somehow from the mainstream of our work. In a first reading, one may skip this part and go directly to Section 4.

It does not make much sense to ask about the cardinality of $\mathcal{T}(E, P, Q)$. As mentioned before, a weak bi-spectrum is typically infinite because it is formed by pieces of smooth curves. It is interesting however to see how many pieces show up if one draws a picture of the weak bi-spectrum. By a *plane algebraic curve* we understand a set of the form

$$\Phi^{-1}(0) = \left\{(s, t) \in \mathbb{R}^2 : \Phi(s, t) = 0\right\},$$

with $\Phi : \mathbb{R}^2 \to \mathbb{R}$ standing for a nonconstant bivariate polynomial. In the sequel, the notation $\mathbb{H}(p, q)$ indicates the set of functions of the form

$$\Phi(s, t) = \sum_{i=0}^{p} \sum_{j=0}^{q} \gamma_{i,j} s^i t^j \tag{24}$$

with $\gamma_{p,q} \neq 0$. A function like (24) is a nonconstant bivariate polynomials of degree p in the first variable and of degree q in the second variable. The *total degree* of (24) is defined as the sum $p + q$ of both degrees. The term $s^p t^q$ is called the *dominant monomial* of (24).

Proposition 3.2. *Let $P \in \Xi(\mathbb{R}^n)$ and $Q \in \Xi(\mathbb{R}^m)$ be polyhedral cones. Then $\mathcal{T}(E, P, Q)$ is contained in a plane algebraic curve. More precisely,*

$$\mathcal{T}(E, P, Q) \subset \left\{(s, t) \in \mathbb{R}^2 : \Phi_{E,P,Q}(s, t) = 0\right\}, \tag{25}$$

where

$$\Phi_{E,P,Q}(s, t) = \prod_{(F,G)} \Phi_E^{F,G}(s, t) \tag{26}$$

is a product running over all pairs $(F, G) \in \mathcal{F}(P) \times \mathcal{F}(Q)$, and $\Phi_E^{F,G}$ is a certain element of $\mathbb{H}(\dim F, \dim G)$.

Proof. For the unconstrained bivariate eigenvalue problem, one clearly has

$$\mathcal{T}(E) \subset \Phi_E^{-1}(0) \tag{27}$$

with $\Phi_E : \mathbb{R}^2 \to \mathbb{R}$ standing for the *bivariate characteristic polynomial* associated to E, i.e.,

$$\Phi_E(s, t) = \det \begin{bmatrix} A - sI_n & B \\ C & D - tI_m \end{bmatrix}.$$

Needless to say, the inclusion (27) can be strict. For the cone-constrained model (2) there is no such thing as an associated bivariate characteristic polynomial. However, thanks to the reduction mechanism (20), each $\mathcal{T}(E, \mathrm{span}F, \mathrm{span}G)$ is contained in a plane algebraic curve. Indeed, one has

$$\mathcal{T}(E, \mathrm{span}\, F, \mathrm{span}\, G) \subset \left[\Phi_E^{F,G}\right]^{-1}(0),$$

where $\Phi_E^{F,G} \in \mathbb{H}(\dim F, \dim G)$ is the bivariate characteristic polynomial associated to the block structured matrix (21). Of course, $R \in \mathbb{M}_{n,\dim F}$ and $S \in \mathbb{M}_{m,\dim G}$ are full rank matrices such that

$$\mathrm{span}\, F = \mathrm{Im}\, R, \quad \mathrm{span}\, G = \mathrm{Im}\, S.$$

This argument applies to each pair (F, G). By taking (22) into account, one sees that $\mathcal{T}(E, P, Q)$ is contained in the plane algebraic curve

$$[\Phi_{E,P,Q}]^{-1}(0) = \bigcup_{F \in \mathcal{F}(P),\, G \in \mathcal{F}(Q)} \left[\Phi_E^{F,G}\right]^{-1}(0),$$

which is precisely the right-hand side of (25). □

Let $f_P(k)$ denote the number of k-dimensional faces of P. One may see $f_P(\cdot)$ as a sort of discrete density function. If one writes

$$f_P = \sum_{k=1}^{\dim P} f_P(k), \quad \hat{f}_P = \sum_{k=1}^{\dim P} k f_P(k),$$

then the first sum corresponds to the number of nonzero faces of P, whereas the second sum looks like a mathematical expectation. The symbols f_Q and $\hat{f}_Q$ are defined in a similar way. It is interesting to observe that the product in (26) has $f_P f_Q$ factors. When one works with a prescribed matrix E, one can drop all the terms $\Phi_E^{F,G}$ that do not vanish on $\mathbb{R}^2$. Another interesting observation is that one can compute both degrees of the bivariate polynomial $\Phi_{E,P,Q}$. Indeed, one obtains

$$\Phi_{E,P,Q} \in \mathbb{H}(f_Q \hat{f}_P, f_P \hat{f}_Q)$$

by developing the product

$$\begin{aligned}
\prod_{(F,G)} s^{\dim F} t^{\dim G} &= \left(\prod_F s^{\dim F}\right)^{f_Q} \left(\prod_G t^{\dim G}\right)^{f_P} \\
&= \left(\prod_{k=1}^{\dim F} \prod_{\dim F = k} s^k\right)^{f_Q} \left(\prod_{\ell=1}^{\dim Q} \prod_{\dim G = \ell} t^\ell\right)^{f_P} \\
&= \left(s^{\hat{f}_P}\right)^{f_Q} \left(t^{\hat{f}_Q}\right)^{f_P}.
\end{aligned}$$

The term $s^{\dim F} t^{\dim G}$ is of course the dominant monomial of $\Phi_E^{F,G}$.

Example 3.3. Let $P = \mathbb{R}^n_+$ be the Pareto cone in $\mathbb{R}^n$ and Q be a half-space in $\mathbb{R}^m$. Then

$$f_P(k) = \frac{n!}{k!(n-k)!}, \quad f_P = 2^n - 1, \quad \hat{f}_P = n2^{n-1}.$$

On the other hand, Q has one face of dimension m and one face of dimension $m - 1$. Thus, $f_Q = 2$ and $\hat{f}_Q = 2m - 1$. For any matrix E, the bivariate polynomial $\Phi_{E,P,Q}$ is of degree $n2^n$ in the first variable and of degree $(2^n - 1)(2m - 1)$ in the second variable.

Proposition 3.2 is a localization result: it tells where the weak bi-eigenvalues of (2) are to be sought. The next corollary is a cardinality result: it tells how many times an oblique line can intersect a weak bi-spectrum. A line in the plane is *oblique* if it is neither vertical nor horizontal.

Corollary 3.4. *If $P \in \Xi(\mathbb{R}^n)$ and $Q \in \Xi(\mathbb{R}^m)$ are polyhedral cones, then an oblique line in $\mathbb{R}^2$ intersects $\mathcal{T}(E, P, Q)$ in at most*

$$f_{P,Q} := f_Q \hat{f}_P + f_P \hat{f}_Q = f_P f_Q \left[\frac{\hat{f}_P}{f_P} + \frac{\hat{f}_Q}{f_Q}\right] \tag{28}$$

points. The upper bound (28), which depends only on the facial structure of P and Q, is uniform with respect to the matrix E.

Proof. An oblique line is a set of the form $\{(\alpha_1 + r\beta_1, \alpha_2 + r\beta_2) : r \in \mathbb{R}\}$ with $\beta_1 \neq 0$ and $\beta_2 \neq 0$. The univariate polynomial

$$r \in \mathbb{R} \mapsto \varphi(r) = \Phi_{E,P,Q}(\alpha_1 + r\beta_1, \alpha_2 + r\beta_2)$$

is nonconstant and its degree is equal to $f_{P,Q}$. Hence, φ has at most $f_{P,Q}$ real roots. □

One can construct nonpolyhedral cones $P \in \Xi(\mathbb{R}^n)$ and $Q \in \Xi(\mathbb{R}^m)$ and a matrix E such that some oblique line intersects $\mathcal{S}(E, P, Q)$ infinitely many times. So, in a nonpolyhedral setting, it is not just the weak bi-spectrum, but also the strong bi-spectrum that could fail to satisfy the finite intersection property on oblique lines.

4. Nonnegativity constraints

4.1. Interiority and binding conditions

Perhaps the most important example of cone-constrained bivariate eigenvalue problem is

$$\begin{cases} 0 \leqslant x \perp (Ax + By - \lambda x) \geqslant 0, \\ 0 \leqslant y \perp (Cx + Dy - \mu y) \geqslant 0 \end{cases} \tag{29}$$

with "$\geqslant$" standing for usual componentwise ordering. This corresponds to the equilibrium model (2) when $P = \mathbb{R}^n_+$ and $Q = \mathbb{R}^m_+$, that is, when the state vectors x and y are required to have nonnegative entries. We use the notation

$$\mathcal{T}_+(E) = \mathcal{T}\left(E, R^n_+, \mathbb{R}^m_+\right),$$

$$\mathcal{S}_+(E) = \mathcal{S}\left(E, R^n_+, \mathbb{R}^m_+\right)$$

for indicating the weak and strong bi-spectra of (29).

The existence of weak bi-eigenvalues for (29) is taken care by Theorem 2.6, so we do not have to worry about that. The next theorem fully characterizes the weak bi-spectrum of (29). Some comments on notation are in order. In the same way as $m_{i,j}$ indicates the (i,j)-entry of a rectangular matrix M, the symbol $M^{I,J}$ refers to the principal submatrix of M formed with the rows indexed by I and columns indexed by J. We introduce also the symbol $\langle n \rangle = \{1, \ldots, n\}$.

Theorem 4.1. *For a pair $(\lambda, \mu) \in \mathbb{R}^2$ the following statements are equivalent*:

(a) *The nonnegatively constrained system* (29) *has a solution (x, y) in $\mathbb{T}_{n,m}$.*
(b) *There are index sets $I \subset \langle n \rangle$ and $J \subset \langle m \rangle$ such that*

$$\begin{cases} A^{I,I}\xi + B^{I,J}\eta = \lambda\,\xi \\ C^{J,I}\xi + D^{J,J}\eta = \mu\,\eta \end{cases} \tag{30}$$

has a solution $(\xi, \eta) \in \mathbb{R}^{|I|+|J|}$ satisfying the "interiority" conditions

$$\xi \in \operatorname{int}\left(\mathbb{R}^{|I|}_+\right), \quad \eta \in \operatorname{int}\left(\mathbb{R}^{|J|}_+\right) \tag{31}$$

and the "binding" conditions

$$\sum_{k \in I} a_{i,k}\xi_k + \sum_{\ell \in J} b_{i,\ell}\eta_\ell \geqslant 0 \quad \text{for all } i \in \langle n \rangle \backslash I, \tag{32}$$

$$\sum_{k \in I} c_{j,k}\xi_k + \sum_{\ell \in J} d_{j,\ell}\eta_\ell \geqslant 0 \quad \text{for all } j \in \langle m \rangle \backslash J. \tag{33}$$

Furthermore, when these equivalent statements hold, (x, y) can be constructed from (ξ, η) by setting

$$x_i = \begin{cases} \xi_i & \text{if } i \in I, \\ 0 & \text{if } i \in \langle n \rangle \backslash I, \end{cases} \quad \text{and} \quad y_j = \begin{cases} \eta_j & \text{if } j \in J, \\ 0 & \text{if } j \in \langle m \rangle \backslash J. \end{cases} \tag{34}$$

Proof. A result of this kind is stated in [29, Theorem 4.1] for a single linear complementary problem. The proof is now slightly more complicated, but the general pattern is similar. One starts by reformulating (29) in a componentwise manner:

$$x_i \geqslant 0, \tag{35}$$

$$(Ax + By - \lambda x)_i \geqslant 0 \quad \text{for all } i \in \langle n \rangle, \tag{36}$$

$$x_i(Ax + By - \lambda x)_i = 0, \tag{37}$$

$$y_j \geqslant 0, \tag{38}$$

$$(Cx + Dy - \mu y)_j \geqslant 0 \quad \text{for all } j \in \langle m \rangle, \tag{39}$$

$$y_j(Cx + Dy - \mu y)_j = 0. \tag{40}$$

Let (x, y) be as in (a). We introduce the index sets

$$I = \{i \in \langle n \rangle : x_i > 0\}, \quad J = \{j \in \langle m \rangle : y_j > 0\}$$

and the positive variables $\xi_i = x_i$ for all $i \in I$ and $\eta_j = y_j$ for all $j \in J$. If one uses (37) and (40), then one gets

$$\sum_{k \in I} a_{i,k}\xi_k + \sum_{\ell \in J} b_{i,\ell}\eta_\ell = \lambda \xi_i \quad \text{for all } i \in I,$$

$$\sum_{k \in I} c_{j,k}\xi_k + \sum_{\ell \in J} d_{j,\ell}\eta_\ell = \mu \eta_j \quad \text{for all } j \in J,$$

respectively. The matrix format of this system is precisely (30). The conditions (32) and (33) are obtained by working out (36) and (38), respectively. One can also proceed the other way around. One starts with I, J, ξ, η as in (b) and shows that (34) yields a pair (x, y) as in (a). This is the basic idea of the proof, the details are not worth dwelling on. $\square$

Remark 4.2. If (x, y) and (ξ, η) are related by (34), then $\|x\| = \|\xi\|$ and $\|y\| = \|\eta\|$. Hence, an analogous theorem can be stated for characterizing the strong bi-spectrum of (29). It suffices to include the double normalization condition $\|\xi\| = 1$, $\|\eta\| = 1$ in the formulation of (b).

Observe that (30) looks very much like a classical bi-eigenvalue problem, except for the fact that the state vectors ξ and η must have positive entries and, in addition, they must comply to a certain number of binding conditions.

4.2. Detecting bi-copositivity

We end this work with an application to the detection of bi-copositivity. This topic is not meant to be our principal motivation for studying bi-spectra, but it nicely illustrates how Theorem 4.1 can enter into action.

Recall that a symmetric matrix A is declared *copositive* if $x \geqslant 0$ implies $x^T Ax \geqslant 0$. This amounts to saying that the infimal value

$$\kappa(A) = \min\{x^T Ax : x \geqslant 0, \ \|x\| = 1\}$$

is nonnegative. One may see $\kappa(A)$ as a coefficient measuring the degree of copositivity of A. Copositivity has been a very active field of research in the last decade. Its bivariate version reads as follows:

Definition 4.3. The symmetric block structured matrix E is bi-copositive if the quadratic form

$$(x, y) \in \mathbb{R}^{n+m} \mapsto q_E(x, y) = \begin{bmatrix} x \\ y \end{bmatrix}^T E \begin{bmatrix} x \\ y \end{bmatrix}$$

takes nonnegative values when $x \geqslant 0, y \geqslant 0, \|x\| = 1, \|y\| = 1$.

A word of caution is immediately in order: that a direct sum

$$A \oplus D = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

is bi-copositive does not mean that the blocks A and D are both copositive; it just means that

$$\kappa(A) + \kappa(D) \geqslant 0.$$

In other words, the lack of copositivity in one block can be compensated by a high degree of copositivity in the other block.

In general, a necessary condition for bi-copositivity is that

$$a_{i,i} + 2b_{i,j} + d_{j,j} \geqslant 0 \quad \text{for all } i \in \langle n \rangle,\ j \in \langle m \rangle. \tag{41}$$

This can be seen by evaluating the quadratic form q_E at the generators of the Pareto cones. More precisely, one takes x as the ith canonical vector of $\mathbb{R}^n$, and y as the jth canonical vector of $\mathbb{R}^m$. However, (41) is far from being sufficient to ensure bi-copositivity. Here is where strong bi-spectra enter into the picture.

Proposition 4.4. *A necessary and sufficient condition for a symmetric block structured matrix E to be bi-copositive it that*

$$\lambda + \mu \geqslant 0 \quad \textit{for all } (\lambda, \mu) \in \mathcal{S}_+(E). \tag{42}$$

Proof. This follows from Proposition 2.2. $\square$

By way of illustration, let us have a look at the symmetric matrices

$$E = \begin{bmatrix} 5 & 5 & -10 & 10 \\ 5 & 5 & -11 & 5 \\ -10 & -11 & 18 & -5 \\ 10 & 5 & -5 & -5 \end{bmatrix}, \quad E' = \begin{bmatrix} 5 & 0 & -4 & 1 \\ 0 & 6 & 0 & 0 \\ -4 & 0 & 2 & 2 \\ 1 & 0 & 2 & 1 \end{bmatrix}, \tag{43}$$

with $n = 2$ and $m = 2$. The second matrix in (43) is not bi-copositive because (41) is violated when $i = 1$ and $j = 1$. Indeed,

$$a_{1,1} + 2b_{1,1} + d_{1,1} = 5 + 2(-4) + 2 < 0.$$

The case of the first matrix in (43) is more involved because all the inequalities in (41) are satisfied. Let us check then the spectrality condition (42). As explained in Theorem 4.1 and Remark 4.2, an exhaustive knowledge of $\mathcal{S}_+(E)$ involves the analysis of nine classical bi-eigenvalue problems. This is because there are $(2^2 - 1)(2^2 - 1) = 9$ ways of choosing the pair (I, J). Consider, for instance, the choice $I = \{1, 2\}$, $J = \{1\}$. The bi-eigenvalue problem

$$\begin{aligned} 5\xi_1 + 5\xi_2 - 10\eta_1 &= \lambda\xi_1, \\ 5\xi_1 + 5\xi_2 - 11\eta_1 &= \lambda\xi_2, \\ -10\xi_1 - 11\xi_2 + 18\eta_1 &= \mu\eta_1 \end{aligned}$$

complemented with

$$\begin{aligned} &\xi_1 > 0, \quad \xi_2 > 0, \quad \eta_1 > 1 && \text{(interiority)}, \\ &\xi_1^2 + \xi_2^2 = 1, \quad \eta_1^2 = 1 && \text{(double normalization)}, \\ &10\xi_1 + 5\xi_2 - 5\eta_1 \geqslant 0 && \text{(binding condition)} \end{aligned}$$

yields $\xi_1 = 3/5, \xi_2 = 4/5, \eta_1 = 1, \lambda = -5$, and $\mu = 16/5$. Thus, we have found a pair $(\lambda, \mu) \in \mathcal{S}_+(E)$ with $\lambda + \mu < 0$. In conclusion, E is not bi-copositive.

4.3. Algorithmic considerations

Both referees raised the question on how to compute numerically a strong bi-eigenvalue. Although our original plan was to deal with this issue in a subsequent publication, we shall briefly discuss here a Newton-type algorithm for the model (29). We get inspiration from the recent work by Adly and Seeger [1]. These authors treat only the case of a single linear complementarity problem, but their ideas extend to a coupled system.

Newton's method in a nonsmooth setting

For the sake of completeness, we recall the formulation of Newton's method in a nonsmooth setting. The purpose of the so-called Semismooth Newton Method (SNM) is finding the roots of a locally Lipschitz vector function, say $h : \mathbb{R}^d \to \mathbb{R}^d$. By Rademacher's theorem, the local Lipschitzness assumption ensures the existence of the Jacobian matrix $h'(z)$ for almost all $z \in \mathbb{R}^d$. Hence, at any reference point z, the Clarke generalized Jacobian

$$\partial h(z) = \mathrm{co}\left\{ M \in \mathbb{M}_d : M = \lim_{k\to\infty} h'(z^k) \text{ for some } \{z^k\}_{k\in\mathbb{N}} \to z \text{ with } z^k \in D_h \right\}$$

is a nonempty compact convex set (cf. [11]). Here, D_h denotes the set of differentiability points of h.

The standard formulation of the SNM reads as follows:

- *Initialization*. Choose an initial point z^0 and set $t = 0$.
- *Iteration*. One has a current point z^t. Choose $M^t \in \partial h(z^t)$ and compute d^t by solving the linear system
$$M^t d^t = -h(z^t). \tag{44}$$
Set $z^{t+1} = z^t + d^t$ and increment t by one.

This algorithm has been studied in depth by many authors. Some additional assumptions on h are needed for ensuring that (44) admits a unique solution and that $\{z^t\}_{t\in\mathbb{N}}$ converges. The theorem below is taken from Qi and Sun [27, Section 3].

Theorem 4.5. *Let $\bar{z}$ be a root of a locally Lipschitz function $h : \mathbb{R}^d \to \mathbb{R}^d$. Suppose that h is semismooth at $\bar{z}$, and that every matrix in $\partial h(\bar{z})$ is nonsingular. Then there exists a neighborhood V of $\bar{z}$ such that the SNM initialized at any $z^0 \in V$ generates a sequence $\{z^t\}_{t\in\mathbb{N}}$ that converges superlinearly to $\bar{z}$.*

We opted for a formulation of Theorem 4.5 that is as simple as possible, but the specialized literature in the area provides also conditions guaranteeing a quadratic rate of convergence of the sequence $\{z^t\}_{t\in\mathbb{N}}$. The notion of semismoothness has been widely used in the last two decades and does not need further presentation.

Finding strong bi-eigenvalues

We are now ready to address the question of finding a strong bi-eigenvalue for the model (29). The problem at hand amounts to finding a solution $z = (x, u, y, v, \lambda, \mu)$ to the system

$$\begin{aligned}
x \geqslant 0, \quad y \geqslant 0 &\quad \text{primal feasibility,}\\
u \geqslant 0, \quad v \geqslant 0 &\quad \text{dual feasibility,}\\
u^T x = 0, \quad v^T y = 0 &\quad \text{complementarity slackness,}\\
Ax + By - \lambda x = u,\ Cx + Dy - \mu y = v &\quad \text{stationarity,}\\
\|x\| = 1, \quad \|y\| = 1 &\quad \text{double normalization.}
\end{aligned}$$

In fact, we reformulate this as a system of $d = 2(n + m + 1)$ equations in the same number of variables:

$$\mathcal{U}_\varphi(x, u) = 0, \tag{45}$$
$$\mathcal{V}_\varphi(y, v) = 0, \tag{46}$$
$$Ax + By - \lambda x - u = 0, \tag{47}$$
$$Cx + Dy - \mu y - v = 0, \tag{48}$$
$$\|x\|^2 - 1 = 0, \tag{49}$$
$$\|y\|^2 - 1 = 0. \tag{50}$$

Here $\mathcal{U}_\varphi : \mathbb{R}^{2n} \to \mathbb{R}^n$ and $\mathcal{V}_\varphi : \mathbb{R}^{2m} \to \mathbb{R}^m$ are given by

$$\mathcal{U}_\varphi(x, u) = \begin{bmatrix} \varphi(x_1, u_1) \\ \vdots \\ \varphi(x_n, u_n) \end{bmatrix}, \quad \mathcal{V}_\varphi(y, v) = \begin{bmatrix} \varphi(y_1, v_1) \\ \vdots \\ \varphi(y_m, v_m) \end{bmatrix},$$

and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a *complementarity function*, i.e., $\varphi(a, b) = 0$ if and only if $a \geqslant 0$, $b \geqslant 0$, and $ab = 0$. As example of complementarity function, one may consider

$$\varphi(a, b) = \min\{a, b\},$$
$$\varphi(a, b) = a + b - \sqrt{a^2 + b^2}.$$

With any of these two choices, the function $h : \mathbb{R}^d \to \mathbb{R}^d$ associated to (49) and (50) turns out to be locally Lipschitz and semismooth. A formula for the Clarke generalized Jacobian of such function h can be derived as in [1, Lemma 3]. The SNM is then directly applicable to the system (49) and (50). Writing down the details is mere routine and it is space consuming.

Remark 4.6. The numerical experiments reported in [1] concern only the case of a single linear complementarity problem, but they speak very favorable of the SNM as a strategy for solving cone-constrained spectral problems. The SNM can also be applied to the model (2), at least when P and Q are polyhedral cones. A guideline for necessary adjustments can be found in [1, Section 3].

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